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OPTIMUM INVARIANT TESTS IN A CLASS OF MIXED MODELS.

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Introduction.

i) Summary.

A one-way analysis of variance situation with both fixed effects and variance components present is considered. Spjøtvoll's results [3], [4] are used to obtain a locally most powerful test for $\Delta \leq \Delta_0$ against $\Delta > \Delta_0$ (Δ denoting the ratio of occuring variances), and an optimum invariant test for $\Delta \leq \Delta$ against $\Delta = \Delta_1$. The latter depends on Δ_1, but is shown to be a maximin test for $\Delta \leq \Delta_0$ against $\Delta \geq \Delta_1$. Estimation of occuring parameters is discussed, and estimates suggested.

ii) Notation.

$\Sigma(X)$ denotes the covariance matrix of X. $\mathcal{C}(A)$ is the space spanned by the column vectors of the matrix A, A' denotes the transpose, A^{-1} the inverse matrix. I is the identity matrix.

$N(\xi,\sigma^2)$ denotes the normal distribution with mean ξ and variance σ^2, χ^2_ν the chi-square- and T_ν the Student distribution with ν degrees of freedom. $F_{m,n}$ is the Fisher distribution with m and n degrees of freedom.

I. An Example.

1. Definition of the model, and transformation to a canonical form.

The random variable X_{ij} is defined by

$$(1.1) \qquad X_{ij} = \mu + \beta t_{ij} + U_i + V_{ij}, \qquad \begin{array}{ll} j = 1,\dots,n_i & (n_i \geq 2) \\ i = 1,\dots,r & (r \geq 2) \end{array}$$

where μ and β are unknown parameters and all t_{ij}'s are known constants. The U_i's and V_{ij}'s are all independent and have normal distributions with mean zero and variances

$$(1.2) \qquad \begin{array}{l} \operatorname{Var} U_i = \lambda^2 \\ \operatorname{Var} V_{ij} = \sigma^2 \end{array} \qquad j = 1,\dots,n_i; \quad i = 1,\dots,r.$$

Let $\Delta = \frac{\lambda^2}{\sigma^2}$. A test for

$H : \Delta \leq \Delta_o$ against $\Delta > \Delta_o$

is wanted. We first transform the model to a canonical form. Let

$$(1.3) \qquad X_i = [X_{i1},\dots,X_{in_i}]', \qquad i = 1,\dots,r,$$

and define $Y_i = [Y_{i1},\dots,Y_{in_i}]'$ by

$$(1.4) \qquad Y_i = P_i X_i, \qquad i = 1,\dots,r,$$

where P_i is a $n_i x n_i$ orthogonal matrix with the element $p_{jk}^{(i)}$ in place (j,k). Let

$$(1.5) \qquad p_{1k}^{(i)} = \frac{1}{\sqrt{n_i}}, \quad k = 1,\ldots,n_i; \quad i = 1,\ldots,r,$$

and when not $t_{i1} = \ldots = t_{in_i}$

$$(1.6) \qquad p_{2k}^{(i)} = \frac{t_{ik} - \bar{t}_{i.}}{\sqrt{M_i}}, \quad k = 1,\ldots,n_i; \quad i = 1,\ldots,r.$$

Here

$$(1.7) \qquad \left.\begin{aligned} \bar{t}_{i.} &= \frac{1}{n_i}\sum_{j=1}^{n_i} t_{ij} \\ M_i &= \sum_{j=1}^{n_i} (t_{ij}-\bar{t}_{i.})^2 \end{aligned}\right\} i = 1,\ldots,r.$$

Since P_i is orthogonal

$$(1.8) \qquad \left.\begin{aligned} \sum_{k=1}^{n_i} p_{jk}^{(i)} &= 0, \quad j = 2,\ldots,n_i \\ \sum_{k=1}^{n_i} t_{ik}p_{jk}^{(i)} &= 0, \quad j = 3,\ldots,n_i \end{aligned}\right\} i = 1,\ldots,r$$

Hence

$$(1.9) \qquad \left.\begin{aligned} EY_{i1} &= \sqrt{n_i}(\mu+\beta\bar{t}_{i.}) \\ EY_{i2} &= \sqrt{M_i}\,\beta \\ EY_{ij} &= 0, \quad j = 3,\ldots,n_i \end{aligned}\right\} i = 1,\ldots,r.$$

Let B_i denote a $n_i \times n_i$ matrix with all elements equal to 1. Then

$$\Sigma(X_i) = (B_i\Delta + I)\sigma^2 ,$$

and hence

$$\Sigma(Y_i) = (P_iB_iP_i'\Delta + I)\sigma^2 .$$

$P_iB_iP_i'$ has in place (j,k) the element

$$\sum_{m=1}^{n_i} p_{jm}^{(i)} \quad \sum_{m=1}^{n_i} p_{km}^{(i)} .$$

From (1.8) and the fact that $\sum_{k=1}^{n_i} p_{1k}^{(i)} = \sqrt{n_i}$ it follows that the elements of Y_i are all independent with variances

$$\text{(1.10)} \qquad \left.\begin{aligned} &\operatorname{Var} Y_{i1} = (n_i\Delta+1)\sigma^2 \\ &\operatorname{Var} Y_{ij} = \sigma^2, \quad j = 2,\ldots,n_i \end{aligned}\right\} \quad i = 1,\ldots,r$$

Clearly the vectors Y_i are independent, and hence all Y_{ij}, $j = 1,\ldots,n_i$; $i = 1,\ldots,r$ are independent and normally distributed with means and variances given by (1.9) and (1.10).

2. A maximal invariant and its distribution.

It proves helpful to introduce the following transformations

$$(2.1)\qquad T_p = \begin{bmatrix} T_{p1} \\ \vdots \\ T_{pr} \end{bmatrix} = Q_p \begin{bmatrix} Y_{1p} \\ \vdots \\ Y_{rp} \end{bmatrix}, \qquad p = 1,2.$$

Here Q_p $(p = 1,2)$ are orthogonal $r \times r$ matrices with the element $q_{jk}^{(p)}$ in place (j,k). Let

$$(2.2)\qquad \left.\begin{aligned} q_{1k}^{(1)} &= \sqrt{\frac{n_k}{n}} \\ q_{2k}^{(1)} &= \sqrt{\frac{n_k}{M^*}}\,(\bar{t}_{k.}-\bar{t}) \\ q_{1k}^{(2)} &= \sqrt{\frac{M_k}{M}} \end{aligned}\right\} \quad k = 1,\ldots,r,$$

where

$$n = \sum_{i=1}^{r} n_i$$

$$\bar{t} = \frac{1}{n} \sum_{i=1}^{r} \sum_{j=1}^{n_i} t_{ij}$$

$$(2.3)\qquad \begin{aligned} M^* &= \sum_{i=1}^{r} n_i(\bar{t}_{i.}-\bar{t})^2 \\ M &= \sum_{i=1}^{r} M_i = \sum_{i,j} (t_{ij}-\bar{t}_{i.})^2 \end{aligned}$$

(The case with all $\bar{t}_{i.}$'s equal will be treated in Section 7.)

By (2.2)

$$(2.4)\qquad \begin{aligned} ET_{11} &= \sqrt{n}(\mu+\beta\bar{t}), \\ ET_{12} &= \sqrt{M^*}\beta, \\ ET_{1j} &= 0, \quad j = 3,\ldots,r, \end{aligned}$$

and

$$ET_{21} = \sqrt{M}\beta, \tag{2.5}$$
$$ET_{2j} = 0 \;; \quad j = 2,\ldots,r.$$

To find $\Sigma(T_1)$ we note that

$$\Sigma(Y_{11},\ldots,Y_{r1})' = (N\Delta+I)\sigma^2,$$

where N is a $r \times r$ diagonal matrix with n_i in place (i,i). Thus

$$\Sigma(T_1) = (Q_1NQ_1'\Delta+I)\sigma^2 .$$

The elements of T_2 are independent (and independent of T_1) with variances equal to σ^2.

Finally we introduce

$$S_o = [S_{o1},S_{o2}]' = \left[T_{11},T_{12}+\sqrt{\frac{M^*}{M}}T_{21}\right]'$$
$$S_1 = [S_{11},\ldots,S_{1,r-1}]' = \left[T_{12}-\sqrt{\frac{M^*}{M}}T_{21},T_{13},\ldots,T_{1r}\right]'$$
$$S_2 = [S_{21},\ldots,S_{2,n-r-1}]' = \tag{2.6}$$
$$[T_{22},\ldots,T_{2r},Y_{13},\ldots,Y_{1n_1},Y_{23},\ldots,Y_{2n_2},\ldots,Y_{r3},\ldots,Y_{rn_r}]'$$

and we have a one-to-one correspondence between $X' = [X_1',\ldots,X_r']$ and $[S_o',S_1',S_2']$. By (2.4), (2.5) and (1.9)

$$\begin{aligned} &\mathrm{E}S_o = \left[\sqrt{n}(\mu+\beta\bar{t}),\ 2\sqrt{M^*}\beta\right]', \\ (2.7)\qquad &\mathrm{E}S_1 = 0, \\ &\mathrm{E}S_2 = 0. \end{aligned}$$

S_1 and S_2 are independent, and

$$\begin{aligned} (2.8)\qquad &\Sigma(S_1) = \left[Q_1^* N Q_1^{*'} \Delta + I + \frac{M^*}{M} A\right]\sigma^2, \\ &\Sigma(S_2) = I\sigma^2, \end{aligned}$$

where Q_1^* is obtained by deleting the first row of Q_1, and A is the $(r-1)\times(r-1)$ matrix with one in place $(1,1)$ and zeros elsewhere.

For a hypothesis concerning Δ, the situation will remain invariant under the following group G_1 of transformations

$$\begin{aligned} &S_o^* = S_o + [c_1, c_2]', \quad -\infty < c_i < \infty, \quad i = 1,2, \\ (2.9)\qquad &S_1^* = S_1, \\ &S_2^* = S_2. \end{aligned}$$

$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ is a maximal invariant with respect to G_1. Define $C(\Delta)$ by

$$\Sigma(S_1) = C(\Delta)\sigma^2,$$

then from (2.7) and (2.8) the probability density function of S is

$$(2.10)\qquad k(\sigma,\Delta)\cdot\exp\left\{-\frac{1}{2\sigma^2}\left[s_1' C(\Delta)^{-1} s_1 + \sum_{j=1}^{n-r-1} s_{2j}^{\,2}\right]\right\}.$$

Here $k(\sigma,\Delta)$ is a constant depending on σ and Δ only. The distribution of S may therefore be written

$$(2.11) \qquad dP^S_{\Delta,\sigma}(s) = a(s,\Delta,\sigma)b(u(s),\Delta,\sigma)d\mu(s),$$

where μ is the Lebesgues measure, and

$$(2.12) \qquad \begin{aligned} a(s,\Delta,\sigma) &= \exp\left\{-\frac{1}{2\sigma^2}\left[s_1'C(\Delta)^{-1}s_1 - s_1'C(\Delta_o)^{-1}s_1\right]\right\}, \\ u = u(s) &= s_1'C(\Delta_c)^{-1}s_1 + \sum_{j=1}^{n-r-1} s_{2j}^2 , \\ b(u(s),\Delta,\sigma) &= k(\sigma,\Delta)\exp\left\{-\frac{1}{2\sigma^2}u(s)\right\} . \end{aligned}$$

(2.11) together with the parameter-space $\Omega = \{(\Delta,\sigma);\ 0 \leq \Delta < \infty ,\ 0 < \sigma < \infty\}$ defines a class of distributions. For any $(\Delta_o, \sigma_o) \in \Omega$ the distribution $dP^S_{\Delta_o,\sigma_o}(s)$ dominates the class, that is

$$dP^S_{\Delta,\sigma}(s) = \left[a(s,\Delta,\sigma)/a(s,\Delta_o,\sigma_o)\right]\cdot\left[b(u(s),\Delta,\sigma)/b(u(s),\Delta_o,\sigma_o)\right] dP^S_{\Delta_o,\sigma_o}(s).$$

Further $U = u(S)$ is sufficient and complete when $\Delta = \Delta_o$.

3. An optimum invariant test.

Theorem (3.1) of [4] is now applicable. For the hypothesis

$$H' : \Delta = \Delta_o \quad \text{against} \quad (\Delta,\sigma) = (\Delta_1,\sigma_1)$$

we may conclude that there exists, among the tests based on S, a most powerful similar test, φ_1, defined by

$$
(3.1)\qquad \varphi_1(S) = \begin{cases} 1 & \text{when } a(S,\Delta_1,\sigma_1)/a(S,\Delta_0,\sigma_0) > c(U) \\ \gamma(U) & \text{when } a(S,\Delta_1,\sigma_1)/a(S,\Delta_0,\sigma_0) = c(U) \\ 0 & \text{when } a(S,\Delta_1,\sigma_1)/a(S,\Delta_0,\sigma_0) < c(U) \end{cases}
$$

Here $c(U)$ and $\gamma(U)$ are determined by

$$E^{S/U}_{\Delta_0}\ \varphi_1(S) = \epsilon \quad \text{for almost all } U.$$

[$E^{S/U}_{\Delta_0}$ indicates conditional expectation, given U, for $\Delta = \Delta_0$.]

Hence $\varphi_1(S) = 1$ when

$$S_1'C(\Delta_0)^{-1}S_1 - S_1'C(\Delta_1)^{-1}S_1 > c'(U)\left[= 2\sigma_1^2 \ln(c(U))\right],$$

or equivalently when

$$(3.2)\qquad W_1 = \frac{S_1'C(\Delta_0)^{-1}S_1 - S_1'C(\Delta_1)^{-1}S_1}{U} > \frac{c'(U)}{U} = c_1(U).$$

The distribution of W_1 does not involve σ^2. According to theorem 2, Chapter 5 of Lehmann [1], W_1 and U are therefore independent when $\Delta = \Delta_0$. Thus $c_1(U) = c_1$ is a constant independent of U.

Since φ_1 does not depend on σ_1, it is uniformly most powerful, similar and invariant (UMPSI) for testing

$$H_0 : \Delta = \Delta_0 \quad \text{against} \quad \Delta = \Delta_1 .$$

To find the distribution of W_1, we proceed as in [3], Section 4. Since the distribution does not involve σ, we put $\sigma = 1$. By (2.8)

$$c(\Delta) = D + E \cdot \Delta$$

where the matrices D and E do not involve Δ. D and E are easily shown to be positive definite. Thus there exists a non-singular matrix K such that

$$(3.3) \qquad \begin{aligned} KDK' &= I, \\ KEK' &= \lambda, \end{aligned}$$

λ being a diagonal-matrix with $\lambda_1, \ldots, \lambda_{r-1}$ as diagonal elements. The λ_i's are the solutions of $|E - \lambda D| = 0$, and are all > 0.

Define R by

$$(3.4) \qquad R = KS_{1}.$$

Then

$$\Sigma(R) = (\Delta\lambda + I)\sigma^2,$$

and it follows (see [3]) that W_1 has the distribution of

$$(3.5) \qquad W_1(\Delta) = \frac{\sum_{i=1}^{r-1} \left(\frac{\Delta\lambda_i + 1}{\Delta_0\lambda_i + 1} - \frac{\Delta\lambda_i + 1}{\Delta_1\lambda_i + 1} \right) N_i^2}{\sum_{i=1}^{r-1} \frac{\Delta\lambda_i + 1}{\Delta_0\lambda_i + 1} N_i^2 + \sum_{i=r}^{n-2} N_i^2} ,$$

where N_i, $i = 1,\ldots,n-2$ are independent and distributed as $N(0,1)$. The power function of φ_1 is

$$\beta_1(\Delta/\Delta_1) = P\left\{\left(\sum_{i=r}^{n-2} N_i^2\right)^{-1} \sum_{i=1}^{r-1} \frac{\Delta\lambda_i+1}{\Delta_0\lambda_i+1}\left(\frac{(\Delta_1-\Delta_0)\lambda_i}{\Delta_1\lambda_i+1} - c_1\right)N_i^2 > c_1\right\}.$$

From [3], Section 5 we have that $\beta_1(\Delta/\Delta_1)$ is an increasing function of Δ. Thus φ_1 is UMPSI for

$$H_1 : \Delta \leq \Delta_0 \quad \text{against} \quad \Delta = \Delta_1 .$$

This hypothesis problem is also invariant with respect to the group G_2 consisting of all orthogonal-transformations of S_2. A maximal invariant with respect to the group (G_1,G_2), generated by G_1 and G_2, is $(S_1', \sum_{j=1}^{n-r-1} S_{2j}^2)$. Finally the problem is invariant with respect to the following group G_3

$$S^* = k\,S \qquad -\infty < k < \infty .$$

A maximal invariant with respect to $G = (G_1,G_2,G_3)$ is

$$\left[\frac{S_{11}}{\left(\sum_{j=1}^{n-r-1} S_{2j}^2\right)^{\frac{1}{2}}}, \frac{S_{12}}{\left(\sum_{j=1}^{n-r-1} S_{2j}^2\right)^{\frac{1}{2}}}, \ldots, \frac{S_{1,r-1}}{\left(\sum_{j=1}^{n-r-1} S_{2j}^2\right)^{\frac{1}{2}}}\right].$$

The distribution of this statistic involves Δ only. Therefore a test that is invariant with respect to G, is similar on Δ_0. Thus the UMPSI test φ_1 also is UMPI.

The group G satisfies the conditions of the Hunt-

Stein theorem ([1], page 336), and so φ_1 is a maximin test for H_1. Since $\beta_1(\Delta/\Delta_1)$ is increasing in Δ, φ_1 is a maximin test for

$$H_2 : \Delta \leq \Delta_o \quad \text{against} \quad \Delta \geq \Delta_1 .$$

Letting $\Delta_1 \to \infty$ in (3.2) the following test for $H : \Delta \leq \Delta_o$ against $\Delta > \Delta_o$ is obtained (see [3], Section 6).

$$(3.6) \quad \varphi_2(S) = \begin{cases} 1 & \text{when } W_2 = \dfrac{S_1' C(\Delta_o)^{-1} S_1}{\sum_{j=1}^{n-r-1} S_{2j}^2} > c_2 \quad \text{(constant)} \\[2ex] 0 & \text{when } W_2 < c_2 \end{cases}$$

When $\Delta = \Delta_o$, $\frac{n-r-1}{r-1} W_2$ is distributed as $F_{r-1,n-r-1}$.

Generally we may base a test for H on

$$\frac{\sum_{i=1}^{r-1} k_i R_i^2}{\sum_{j=1}^{n-r-1} S_{2j}^2} ,$$

where the R_i's are the elements of R, and the k_i's are constants. Approximating $\Sigma k_i R_i^2$ with a χ^2-distribution, we find an approximate expression for the power function.

4. A locally most powerful invariant test.

From [4] we have the following definition: A level ϵ test φ_o of $H_3 : \Delta = \Delta_o$ against $\Delta > \Delta_o$, is locally most powerful (LMP) if, given any other level ϵ test φ, there exists for each σ a d such that $\beta(\Delta,\sigma,\varphi_o) \geq \beta(\Delta,\sigma,\varphi)$ when $\Delta_o \leq \Delta < \Delta_o + d$. ($\beta(\Delta,\sigma,\varphi)$ is the power function of φ.)

From theorem 2.9 in [1] it follows that for any test φ for H_3 the differentiation with respect to Δ of $\beta(\Delta,\sigma,\varphi)$ can be carried out under the integral sign.

By theorem 3.2. in [4] then the following test maximizes the derivative of the power function at (σ_o, Δ_o) among similar level ϵ tests based upon S.

$$
(4.1) \quad \varphi_3(S) = \begin{cases} 1 & \text{when } a'_\Delta(S,\Delta_o,\sigma_o)/a(S,\Delta_o,\sigma_o) > c(U) \\ \gamma(U) & \text{when } a'(S,\Delta_o,\sigma_o)/a(S,\Delta_o,\sigma_o) = c(U) \\ 0 & \text{when } a'_\Delta(S,\Delta_o,\sigma_o)/a(S,\Delta_o,\sigma_o) < c(U) \end{cases}
$$

Here a'_Δ denotes the derivative of the function a with respect to Δ, and c(U) and $\gamma(U)$ are determined by $E^{S/U}_{\Delta_o} \varphi_3(S) = \epsilon$ for almost all U.

Letting

$$C^*(\Delta) = - \frac{d}{d\Delta} \left[C(\Delta)^{-1}\right],$$

we have $\varphi_3(S) = 1$ when

$$S_1' C^*(\Delta_o) S_1 > 2\sigma_o^2 \; c(U) = c_o(U),$$

that is when

(4.2) $$W_3 = \frac{S_1'C^*(\Delta_0)S_1}{U} > \frac{c_o(U)}{U} = c_3(U).$$

As in (3.2) $c_3(U) = c_3$ is a constant independent of U.

Since W_3 does not depend upon σ_o, and its distribution does not involve σ, φ_3 is LMPSI for H_3. And since here invariance (with respect to G) implies similarity, φ_3 also is LMPI.

From (3.4) we have

(4.3) $$S_1'C(\Delta)^{-1}S_1 = \sum_{i=1}^{r-1} \frac{R_i^2}{\Delta\lambda_i+1},$$

which yields

$$S_1'C^*(\Delta)S_1 = \sum_{i=1}^{r-1} \frac{\lambda_i}{(\Delta\lambda_i+1)^2} R_i^2.$$

The power function of φ_3 is

(4.4) $$\beta_3(\Delta) = P\left[\frac{\sum_{i=1}^{r-1} \frac{\Delta\lambda_i+1}{\Delta_0\lambda_i+1}\left[\frac{\lambda_i}{\Delta_0\lambda_i+1} - c_3\right]N_i^2}{\sum_{i=r}^{n-2} N_i^2} > c_3\right].$$

As for $\beta_1(\Delta/\Delta_1)$ we see that $\beta_3(\Delta)$ is increasing in Δ. And so φ_3 is LMPI also for H.

5. The test statistics expressed by means of the original observations.

We find

$$(5.1)\qquad Y_{i1} = \frac{1}{\sqrt{n_i}} \sum_{j=1}^{n_i} X_{ij} = \sqrt{n_i}\, \bar{X}_{i.} \ ,$$

$$(5.2)\qquad Y_{i2} = \frac{1}{\sqrt{M_i}} \sum_{j=1}^{n_i} (t_{ij}-\bar{t}_{i.})X_{ij} \ ,$$

$$(5.3)\qquad \begin{aligned} \sum_{i=1}^{r}\sum_{j=3}^{n_i} Y_{ij}^2 &= \sum_{i=1}^{r}\sum_{j=1}^{n_i} X_{ij}^2 - \sum_{i=1}^{r} n_i \bar{X}_{i.}^2 \\ &\quad - \sum_{i=1}^{r} \frac{1}{M_i}\Big[\sum_{j=1}^{n_i} (t_{ij}-\bar{t}_{i.})X_{ij}\Big]^2 \\ &= \sum_{i=1}^{r}\sum_{j=1}^{n_i}\Big[X_{ij}-\bar{X}_{i.} \\ &\quad - \Big(\frac{1}{M_i}\sum_{k=1}^{n_i}(t_{ik}-\bar{t}_{i.})X_{ik}\Big)(t_{ij}-\bar{t}_{i.})\Big]^2 , \end{aligned}$$

$$(5.4)\qquad T_{11} = \frac{1}{\sqrt{n}} \sum_{i=1}^{r} \sqrt{n_i}\, Y_{i1} = \sqrt{n}\, \bar{X} \ ,$$

$$(5.5)\qquad T_{12} = \frac{1}{\sqrt{M^*}} \sum_{i=1}^{r} \sqrt{n_i}(\bar{t}_{i.}-\bar{t})Y_{i1} = \frac{1}{\sqrt{M^*}} \sum_{i=1}^{r} n_i(\bar{t}_{i.}-\bar{t})\bar{X}_{i.} \ ,$$

$$(5.6)\qquad T_{21} = \frac{1}{\sqrt{M}} \sum_{i=1}^{r} \sqrt{M_i}\, Y_{i2} = \frac{1}{\sqrt{M}} \sum_{i,j} (t_{ij}-\bar{t}_{i.})X_{ij} \ ,$$

$$(5.7)\qquad \begin{aligned} \sum_{i=2}^{r} T_{2i}^2 &= \sum_{i=1}^{r} Y_{i2}^2 - T_{21}^2 \ \Big[= \sum_{i=1}^{r} M_i\Big(\frac{Y_{i2}}{\sqrt{M_i}} - \frac{T_{21}}{\sqrt{M}}\Big)^2\Big] \\ &= \sum_{i=1}^{r} \frac{1}{M_i}\Big[\sum_{j=1}^{n_i}(t_{ij}-\bar{t}_{i.})X_{ij}\Big]^2 - \frac{1}{M}\Big[\sum_{i,j}(t_{ij}-\bar{t}_{i.})\, X_{ij}\Big]^2 . \end{aligned}$$

By (5.3) and (5.7)

(5.8)
$$\sum_{j=1}^{n-r-1} S_{2j}^2 = \sum_{i,j} X_{ij}^2 - \sum_{i=1}^{r} n_i \overline{X}_{i.}^2 - \frac{1}{M}\left[\sum_{i,j} (t_{ij}-\overline{t}_{i.})X_{ij}\right]^2$$
$$= \sum_{i,j} \left[X_{ij} - \overline{X}_{i.} - \frac{T_{21}}{\sqrt{M}} (t_{ij}-\overline{t}_{i.})\right]^2 .$$

It now remains to find $S_1' C(\Delta)^{-1} S_1$, and we introduce

(5.9)
$$N_i(\Delta) = \frac{n_i}{n_i\Delta+1}$$
$$N(\Delta) = \sum_{i=1}^{r} N_i(\Delta)$$
$$\overline{t}(\Delta) = \frac{1}{N(\Delta)} \sum_{i=1}^{r} N_i(\Delta)\overline{t}_{i.}$$
$$M^*(\Delta) = \sum_{i=1}^{r} N_i(\Delta)(\overline{t}_{i.}-\overline{t}(\Delta))^2$$

(where $N_i(0) = n_i$, $N(0) = n$, $\overline{t}(0) = \overline{t}$ and $M^*(0) = M^*$). Furthermore q_k is the k'th line of Q_1, and

(5.10)
$$q_1^* = \left[\frac{N_1(\Delta)}{\sqrt{n_1 \cdot N(\Delta)}}, \dots, \frac{N_r(\Delta)}{\sqrt{n_r \cdot N(\Delta)}}\right]$$
$$q_2^* = \left[\frac{N_1(\Delta)(\overline{t}_{1.}-\overline{t}(\Delta))}{\sqrt{n_1(M+M^*(\Delta))}}, \dots, \frac{N_r(\Delta)(\overline{t}_{r.}-\overline{t}(\Delta))}{\sqrt{n_r(M+M^*(\Delta))}}\right]$$

Letting

$$(5.11)\quad S^* = \begin{bmatrix} S_1^* \\ S_2^* \\ S_{11} \\ S_{12} \\ \vdots \\ S_{1,r-1} \end{bmatrix} = \begin{bmatrix} 0 & q_1^* \\ \sqrt{\frac{M}{M+M^*(\Delta)}} & q_2^* \\ -\sqrt{\frac{M^*}{M}} & q_2 \\ 0 & q_3 \\ \vdots & \vdots \\ 0 & q_r \end{bmatrix} \begin{bmatrix} T_{21} \\ Y_{11} \\ Y_{21} \\ Y_{31} \\ \vdots \\ Y_{r1} \end{bmatrix}$$

we get

$$\mathrm{Cov}(S_1^*, S_2^*) = 0,$$

$$\left.\begin{aligned} \mathrm{Cov}(S_1^*, S_{1j}) &= 0 \\ \mathrm{Cov}(S_2^*, S_{1j}) &= 0 \end{aligned}\right\} j = 1,\dots,r-1,$$

$$\mathrm{Var}\ S_1^* = \mathrm{Var}\ S_2^* = \sigma^2$$

Defining $B(\Delta) = \frac{1}{\sigma^2} \sum (S^*)$ we have

$$(5.12)\qquad S^{*\prime} B(\Delta)^{-1} S^* = S_1^{*2} + S_2^{*2} + S_1' C(\Delta)^{-1} S_1,$$

$$(5.13)\qquad S^{*\prime} B(\Delta)^{-1} S^* = T_{21}^2 + \sum_{i=1}^{r} \frac{Y_{i1}^2}{n_i\Delta+1}\ .$$

Furthermore

$$(5.14)\qquad \begin{aligned} S_1^* &= \frac{1}{\sqrt{N(\Delta)}} \sum_{i=1}^{r} \frac{\sqrt{n_i}}{n_i\Delta+1} Y_{i1}, \\ S_2^* &= (M+M^*(\Delta))^{-\frac{1}{2}} \left[\sqrt{M}\, T_{21} + \sum \frac{\sqrt{n_i}}{n_i\Delta+1}(\bar{t}_{i\cdot} - \bar{t}(\Delta)) Y_{i1}\right]. \end{aligned}$$

Introducing

(5.15)

$$\overline{X}(\Delta) = \frac{1}{N(\Delta)} \sum_{i=1}^{r} N_i(\Delta)\overline{X}_{i.}; \quad (\overline{X}(0) = \overline{X})$$

$$\overline{X}^*(\Delta) = \frac{1}{M^*(\Delta)} \sum_{i=1}^{r} N_i(\Delta)(\overline{t}_{i.}-\overline{t}(\Delta))\overline{X}_{i.}; (\overline{X}^*(0) = \frac{T_{12}}{\sqrt{M^*}})$$

into (5.14) we get from (5.12) and (5.13)

(5.16)

$$S_1'C(\Delta)^{-1}S_1 = \sum_{i=1}^{r} N_i(\Delta)(\overline{X}_{i.}-\overline{X}(\Delta))^2 + T_{21}^2$$

$$- \frac{1}{M+M^*(\Delta)}\left[M^*(\Delta)\overline{X}^*(\Delta) + \sqrt{M}\,T_{21}\right]^2 .$$

This may be rewritten as

(5.17)

$$S_1'C(\Delta)^{-1}S_1 = \sum_{i=1}^{r} N_i(\Delta)(\overline{X}_{i.}-\overline{X}(\Delta))^2 - M^*(\Delta)\overline{X}^*(\Delta)^2$$

$$+ \frac{MM^*(\Delta)}{M+M^*(\Delta)}\left[\overline{X}^*(\Delta) - \frac{T_{21}}{\sqrt{M}}\right]^2 =$$

$$\sum_{i=1}^{r} N_i(\Delta)\left[\overline{X}_{i.}-\overline{X}(\Delta)-\overline{X}^*(\Delta)(\overline{t}_{i.}-\overline{t}(\Delta))\right]^2$$

$$+ \frac{MM^*(\Delta)}{M+M^*(\Delta)}\left[\overline{X}^*(\Delta) - \frac{T_{21}}{\sqrt{M}}\right]^2 .$$

(When $\Delta = 0$ the sum in the last expression reduces to $\sum n_i\left[\overline{X}_{i.}-\overline{X} - \frac{1}{\sqrt{M^*}}T_{12}(\overline{t}_{i.}-\overline{t})\right]^2$.)

To find $S_1'C^*(\Delta)S_1 = -\frac{d}{d\Delta}(S_1'C(\Delta)^{-1}S_1)$ we need

$$\frac{d}{d\Delta}\left(\sum_{i=1}^{r} N_i(\Delta)(\overline{X}_{i.}-\overline{X}(\Delta))^2\right) = -\sum_{i=1}^{r} N_i^2(\Delta)(\overline{X}_{i.}-\overline{X}(\Delta))^2,$$

$$\frac{d}{d\Delta}(M^*(\Delta)\overline{X}^*(\Delta)) = -\sum_{i=1}^{r} N_i(\Delta)^2(\overline{X}_{i.}-\overline{X}(\Delta))\,(\overline{t}_{i.}-\overline{t}(\Delta))$$

$$= -B_{x,t}(\Delta),$$

$$\frac{d}{d\Delta}(M^*(\Delta)) = -\sum_{i=1}^{r} N_i(\Delta)^2(\bar{t}_{i.}-\bar{t}(\Delta))^2 = -M_o^*(\Delta).$$

Combining this with (5.16),

$$S_1'C^*(\Delta)S_1 = \sum_{i=1}^{r} N_i(\Delta)^2(\bar{X}_{i.}-\bar{X}(\Delta))^2$$

$$(5.18)\quad - \frac{2}{M+M^*(\Delta)}\left[M^*(\Delta)\bar{X}^*(\Delta)+\sqrt{M}\,T_{21}\right]B_{x,t}(\Delta)$$

$$+ \frac{M_o^*(\Delta)}{(M+M^*(\Delta))^2}\left[M^*(\Delta)\bar{X}^*(\Delta)+\sqrt{M}\,T_{21}\right]^2 .$$

By (5.8) and (5.16)

$$U = S_1'C(\Delta_o)^{-1}S_1 + \sum_{j=1}^{n-r-1} S_{2j}^2 =$$

$$(5.19)\quad \sum_{i=1}^{r} N_i(\Delta_o)(\bar{X}_{i.}-\bar{X}(\Delta_o))^2 - \frac{(M^*(\Delta_o)\bar{X}^*(\Delta_o)+\sqrt{M}T_{21})^2}{M+M^*(\Delta_o)} +$$

$$+ \sum_{i,j} (X_{ij}-\bar{X}_{i.})^2.$$

(When $\Delta_o = 0$, (5.19) reduces to

$$U = \sum_{i,j} X_{ij}^2 - n\,\bar{X}^2 - \frac{1}{M+M^*}\left[\sqrt{M^*}\,T_{12}+\sqrt{M}\,T_{21}\right]^2 ,$$

where

$$\sqrt{M^*}\,T_{12}+\sqrt{M}\,T_{21} = \sum_{i,j} (t_{ij}-\bar{t})X_{ij},$$

$$M+M^* = \sum_{i,j} (t_{ij}-\bar{t})^2 = M_{tot}.)$$

6. The balanced model.

When $n_i = m$, $i = 1,\dots,r$, all Y_{i1} have the same variance $(m\Delta+1)\sigma^2$, and so all T_{1i}, $i = 1,\dots,r$ are independent with variance $(m\Delta+1)\sigma^2$. Furthermore S_{1i}, $i = 1,\dots,r-1$ are independent, and

$$\operatorname{Var} S_{11} = \left(1+\Delta m+\frac{M^*}{M}\right)\sigma^2 ,$$

$$\operatorname{Var} S_{1i} = (1+\Delta m)\sigma^2, \quad i = 2,\dots,r-1.$$

Thus

$$(6.1) \qquad S_1'C(\Delta)^{-1}S_1 = \frac{S_{11}^2}{1+\Delta m+\frac{M^*}{M}} + \sum_{i=2}^{r-1} \frac{S_{1i}^2}{1+\Delta m} ,$$

$$(6.2) \qquad S_1'C^*(\Delta)S_1 = \frac{m}{\left(1+\Delta m+\frac{M^*}{M}\right)^2} S_{11}^2 + \frac{m}{(1+\Delta m)^2} \sum_{i=2}^{r-1} S_{1i}^2 .$$

From (6.1)

$$S_1'C(\Delta)^{-1}S_1 = \frac{\left(T_{12}-\sqrt{\frac{M^*}{M}}\,T_{21}\right)^2}{1+\Delta m+\frac{M^*}{M}} + \frac{\sum_{i=1}^{r} Y_{i1}^2 - T_{11}^2 - T_{12}^2}{1+\Delta m} ,$$

where

$$\sum_{i=1}^{r} Y_{i1}^2 - T_{11}^2 - T_{12}^2 = m \sum_{i=1}^{r} \left[\bar{X}_{i.}-\bar{X} - \frac{\sum_{k=1}^{r} m(\bar{t}_{k.}-\bar{t})\bar{X}_{k.}}{M^*}(\bar{t}_{i.}-\bar{t})\right]^2 .$$

When $\Delta = \Delta_0$, both W_1 and W_3 are distributed like

$$(6.3) \qquad W = \frac{a_1Z_1+a_2Z_2}{Z_1+Z_2+Z_3} ,$$

where Z_i is distributed like $\chi^2_{\nu_i}$, $i = 1,2,3.$ $(\nu_1 = 1, \nu_2 = r-2, \nu_3 = n-r-1)$. In [4] it is shown that W is distributed like Y_1Y_2 where Y_1 and Y_2 are independent, Y_1 has a beta-distribution with $\frac{\nu_1+\nu_2}{2}$ and $\frac{\nu_3}{2}$ degrees of freedom and $\frac{Y_2-a_1}{a_2-a_1}$ has a beta-distribution with $\frac{\nu_2}{2}$ and $\frac{\nu_1}{2}$ degrees of freedom. Letting

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,$$

$$k = \frac{\Gamma\left(\frac{\nu_1+\nu_2+\nu_3}{2}\right)}{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)\Gamma\left(\frac{\nu_3}{2}\right)},$$

$$I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x u^{a-1}(1-u)^{b-1}du,$$

we get (when $0 < a_1 < a_2$)

$$P\left[W \geq c\right] = k \int_{\frac{c}{a_2}}^{\frac{c}{a_1}} y^{\frac{\nu_1+\nu_2}{2}-1}(1-y)^{\frac{\nu_3}{2}-1} I_{\frac{a_2y-c}{y(a_2-a_1)}}\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)dy$$

$$(6.4) \qquad + 1 - I_{\frac{c}{a_1}}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_3}{2}\right) \quad \text{when} \quad c \leq a_1,$$

$$P\left[W \geq c\right] = k \int_{\frac{c}{a_2}}^{1} y^{\frac{\nu_1+\nu_2}{2}-1}(1-y)^{\frac{\nu_3}{2}-1} I_{\frac{a_2y-c}{y(a_2-a_1)}}\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)dy$$

when $a_1 < c \leq a_2$.

By (6.4) also expressions for the power-functions $\beta_1(\Delta/a_1)$, $\beta_2(\Delta)$ and $\beta_3(\Delta)$ is obtained.

Leaving out S_{11}, both φ_1, φ_2 and φ_3 reduce to the test φ_4, which has the rejection region

$$W_4 = \frac{\sum_{i=2}^{r+1} S_{1i}^2}{\sum_{j=1}^{n-r-1} S_{2j}^2} > \text{const.}$$

This test has the advantage that its power function is easily obtained, since $\frac{n-r-1}{r-2} \cdot \frac{W_4}{m\Delta+1}$ is distributed like $F_{r-2,n-r-1}$.

7. Special cases of the model.

We consider two special cases.

1. Assume $t_{i1} = \ldots = t_{in_i}$ for all i. Then $EY_{i2} = 0$, $i = 1,\ldots,r$, and redefining S_1 and S_2 as

$$S_1 = [T_{13},\ldots,T_{1r}]',$$

$$S_2 = [Y_{12},\ldots,Y_{1n_1},\ldots,Y_{r2},\ldots,Y_{rn_r}]',$$

$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ is a maximal invariant with respect to a certain group of transformations (see Section 2). φ_1, φ_2 and φ_3 are as before given by (3.2), (3.6) and (4.2).

Furthermore

$$S_1'C(\Delta)^{-1}S_1 = \sum_{i=1}^{r} N_i(\Delta)\bar{X}_{i.}^2 - N(\Delta)\bar{X}(\Delta)^2 - M^*(\Delta)\bar{X}^*(\Delta)^2$$

$$= \sum_{i=1}^{r} N_i(\Delta)\left[\bar{X}_{i.} - \bar{X}(\Delta) - \bar{X}^*(\Delta)(\bar{t}_{i.} - \bar{t}(\Delta))\right]^2 ,$$

$$S_1'C^*(\Delta)S_1 = \sum_{i=1}^{r} N_i(\Delta)^2(\bar{X}_{i.} - \bar{X}(\Delta))^2 - 2\bar{X}^*(\Delta)B_{x,t}(\Delta)$$

$$+ M_o^*(\Delta)\bar{X}^*(\Delta)^2,$$

$$\sum_{j=1}^{n-r} S_{2j}^2 = \sum_{i,j} X_{ij}^2 - \sum_i n_i\bar{X}_{i.}^2 = \sum_{i,j} (X_{ij} - \bar{X}_{i.})^2.$$

When $n_i = m$, $i = 1,\dots,r$, a UMPI test for H is given by $\phi(S) = 1$ when

$$W_2 = \frac{S_1'C(\Delta_o)^{-1}S_1}{\sum_j S_{2j}^2} = \frac{m\sum_{i=1}^{r}\left[\bar{X}_{i.} - \bar{X} - \frac{\sum_{k=1}^{r} m(\bar{t}_{k.} - \bar{t})\bar{X}_{k.}}{M^*}(\bar{t}_{i.} - \bar{t})\right]^2}{(m\Delta_o + 1)\quad \sum_{i,j}(X_{ij} - \bar{X}_{i.})^2} > c_2$$

where $\frac{n-r}{r-2}\,\frac{m\Delta_o+1}{m\Delta+1}\,W_2$ is distributed like $F_{r-2,n-r}$.

2. Assume $\bar{t}_{1.} = \dots = \bar{t}_{r.}$. Then $EY_{i1} = \sqrt{n_i}(\mu+\beta\bar{t})$, $i = 1,\dots,r$, and the mean vector of $[Y_{11},\dots,Y_{r1}]'$ thus lies in a one-dimensional space. We only specify the first line in the matrix Q_1 (see (2.1)), and get

$$ET_{11} = \sqrt{n}(\mu+\beta\bar{t}),$$

$$ET_{1j} = 0 \qquad j = 2,\dots,r.$$

The maximal invariant $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ is obtained from (2.6), letting $M^* = 0$. Furthermore

$$S_1' C(\Delta)^{-1} S_1 = \sum_{i=1}^{r} N_i(\Delta)(\bar{X}_{i.} - \bar{X}(\Delta))^2,$$

$$S_1' C^*(\Delta) S_1 = \sum_{i=1}^{r} N_i(\Delta)^2 (\bar{X}_{i.} - \bar{X}(\Delta))^2,$$

$$\sum_{j=1}^{n-r-1} S_{2j}^2 = \sum_{i=1}^{r} \sum_{j=2}^{n_i} Y_{ij}^2 - T_{21}^2 \quad .$$

When the model is balanced, the UMPI test for H rejects when

$$W_2 = \frac{\frac{m}{m\Delta_0+1} \sum_{i=1}^{r} (\bar{X}_{i.} - \bar{X})^2}{\sum_{i,j} \left[X_{ij} - \bar{X}_{i.} - \left(\frac{1}{M} \sum_{k,l} (t_{kl} - \bar{t}_{k.}) X_{kl}\right)(t_{ij} - \bar{t}_{i.})\right]^2} > c_2$$

where $\frac{n-r-1}{r-1} \frac{m\Delta_0+1}{m\Delta+1} W_2$ is distributed like $F_{r-1,n-r-1}$.

The case 1. treated above is a special case of the model treated by Torgersen [5]. The balanced models in 1. as well as 2. are special cases of the model treated by Spjøtvoll [2]. According to theorem 6 of Section 6 in [1], the UMPI tests found above, when the model is balanced, must coincide with the UMP unbiased tests found in [2] and [5].

When the conditions of 1 as well as 2 are satisfied (that is all t_{ij} are equal), we have $S_1 = [T_{12}, \ldots, T_{1r}]'$ and S_2, consisting of all Y_{ij} with $j \geq 2$, together form a maximal invariant. This is the model treated in [3].

8. Estimation of the unknown parameters.

We have the following unbiased estimate for σ^2.

$$(8.1)\quad \hat{\sigma}^2 = \frac{1}{n-r-1} \sum_{j=1}^{n-r-1} S_{2j}^2 = \frac{1}{n-r-1}\left[\sum_{i=1}^{r} \sum_{j=3}^{n_i} Y_{ij}^2 + \sum_{i=2}^{r} T_{2i}^2 \right].$$

Since $\frac{n-r-1}{\sigma^2}\hat{\sigma}^2$ is distributed like χ^2_{n-r-1}

$$\operatorname{Var}\hat{\sigma}^2 = \frac{2\sigma^4}{n-r-1} .$$

To estimate τ^2 we start with $\sum_{i=1}^{r-1} k_i R_i^2$, where the k_i's are constants, and the R_i's are the elements of R (see 3.4). Thus $\operatorname{Var} R_i = (\Delta\lambda_i+1)\sigma^2$, and

$$(8.2)\quad \hat{\tau}^2 = \frac{1}{\sum_i k_i\lambda_i}\left[\sum_{i=1}^{r-1} k_i R_i^2 - \sum_{i=1}^{r-1} k_i \cdot \hat{\sigma}^2 \right]$$

is an unbiased estimate for τ^2. Letting all $k_i = 1$, $\hat{\tau}^2$ equals (since $\sum_{i=1}^{r-1} R_i^2 = S_1' C(0)^{-1} S_1$)

$$\hat{\tau}_o^2 = \frac{1}{A}\left[\sum_{i=1}^{r} n_i(\bar{X}_{i.}-\bar{X})^2 + T_{21}^2 - \frac{1}{M+M^*}(\sqrt{M^*}T_{12}+\sqrt{M}T_{21})^2 - (r-1)\hat{\sigma}^2 \right],$$

where $A = \sum \lambda_i = n - \frac{\sum_{i=1}^{r} n_i^2}{n} - \frac{M_o^*}{M+M^*}$; $M_o^* = M_o^*(0)$. From (8.2) $\operatorname{Var}\hat{\tau}^2$ is obtained. - When the model is balanced

$$\operatorname{Var}\hat{\tau}_o^2 = \frac{2}{A^2}\left[\frac{(1+\Delta m+\frac{M^*}{M})^2}{(1+\frac{M^*}{M})^2} + (r-2)(1+\Delta m)^2 + \frac{(r-1)^2}{n-r-1} \right]\sigma^4 .$$

As pointed out in [3], Section 6, $\frac{S_1'C(\Delta)^{-1}S_1}{\sum_j S_{2j}^2}$ may be used to derive a confidence interval for Δ. Letting $f_{\frac{\epsilon}{2}}$ and $f_{1-\frac{\epsilon}{2}}$ be lower and upper $\frac{\epsilon}{2}$-point in $F_{r-1,n-r-1}$, we have

$$(8.3) \qquad P\left[f_{\frac{\epsilon}{2}} < \frac{n-r-1}{r-1} \frac{S_1'C(\Delta)^{-1}S_1}{\sum_j S_{2j}^2} < f_{1-\frac{\epsilon}{2}}\right] = 1-\epsilon \ .$$

By (4.3) $S_1'C(\Delta)^{-1}S_1$ is a decreasing function of Δ, and so (8.3) gives an interval for Δ.

After transforming the X_{ij}'s into T_1, T_2 and Y_{ij}, $j > 2$, only three variables have mean different from zero.

$$(8.4) \qquad \begin{aligned} T_{11} &= \sqrt{n}\,\bar{X} &, \quad ET_{11} &= \sqrt{n}(\mu+\beta\bar{t}), \\ T_{12} &= \frac{1}{\sqrt{M^*}} \sum_{i=1}^{r} n_i(\bar{t}_{i.}-\bar{t})\bar{X}_{i.} &, \quad ET_{12} &= \sqrt{M^*}\beta, \\ T_{21} &= \frac{1}{\sqrt{M}} \sum_{i,j} (t_{ij}-\bar{t}_{i.})X_{ij} &, \quad ET_{21} &= \sqrt{M}\beta. \end{aligned}$$

Thus

$$(8.5) \qquad \begin{aligned} \hat{\beta}_1 &= \frac{1}{\sqrt{M^*}} T_{12}, \\ \hat{\beta}_2 &= \frac{1}{\sqrt{M}} T_{21} \end{aligned}$$

are unbiased (and independent) estimates for β, and it follows that

$$(8.6) \qquad \hat{\beta} = c\hat{\beta}_1 + (1-c)\hat{\beta}_2 \ , \qquad c \in 0,1$$

is an unbiased estimate for β. The variances are given by

$$(8.7) \qquad \begin{aligned} \operatorname{Var}\hat{\beta}_1 &= \frac{1}{M^*}\left[1 + \Delta\frac{M_o^*}{M^*}\right]\sigma^2 , \\ \operatorname{Var}\hat{\beta}_2 &= \frac{1}{M}\sigma^2 . \end{aligned}$$

If Δ is known, $c = \dfrac{\operatorname{Var}\hat{\beta}_2}{\operatorname{Var}\hat{\beta}_1 + \operatorname{Var}\hat{\beta}_2}$ makes $\hat{\beta}$ a minimum variance estimate for β. When $\Delta = 0$, the optimal c is $\dfrac{M^*}{M+M^*}$, which makes β equal to

$$(8.8) \qquad \hat{\beta}_3 = \frac{\sum_{i,j}(t_{ij}-\bar{t})X_{ij}}{M_{tot}} , \qquad M_{tot} = \sum_{i,j}(t_{ij}-\bar{t})^2 .$$

Here

$$\operatorname{Var}\hat{\beta}_3 = \frac{1}{M_{tot}}\left[1 + \Delta\frac{M_o^*}{M_{tot}}\right]\sigma^2$$

Since $M_{tot} = M^* + M$, we have $\operatorname{Var}\hat{\beta}_3 \leq \operatorname{Var}\hat{\beta}_1$ for all Δ and σ^2. Furthermore $\operatorname{Var}\hat{\beta}_3 \leq \operatorname{Var}\hat{\beta}_2$ if and only if

$$\Delta < \frac{M^* M_{tot}}{M\ M_o^*} = \frac{\sum_i n_i(\bar{t}_{i.}-\bar{t})^2 \cdot \sum_{i,j}(t_{ij}-\bar{t})^2}{\sum_{i,j}(t_{ij}-\bar{t}_{i.})^2 \cdot \sum_i n_i^2(\bar{t}_{i.}-\bar{t})^2} .$$

When not $t_{i1} = \ldots = t_{in_i}$, all i, hypotheses about β may be tested, since

$$W = \frac{\sqrt{M}(\hat{\beta}_2 - \beta)}{\sqrt{\hat{\sigma}_2}}$$

is distributed like T_{n-r-1}.

To estimate μ we have

$$\hat{\mu} = \frac{1}{\sqrt{n}} T_{11} - \bar{t}\hat{\beta} \qquad (= \bar{X} - \bar{t}\hat{\beta}) \ , \tag{8.9}$$

where

$$\operatorname{Var} \hat{\mu} = \frac{1}{n} \operatorname{Var} T_{11} + \bar{t}^2 \operatorname{Var} \hat{\beta} - \frac{2\bar{t}}{\sqrt{n}} \operatorname{Cov}(T_{11}, \hat{\beta}) \ ,$$

$$\begin{aligned} \operatorname{Cov}(T_{11}, \hat{\beta}) &= \frac{c}{M^* \sqrt{n}} \sum_{i=1}^{r} n_i^2 (\bar{t}_{i.} - \bar{t}) \operatorname{Var} \bar{X}_{i.} \\ &= \frac{c}{M^* \sqrt{n}} \sum_{i=1}^{r} n_i^2 (\bar{t}_{i.} - \bar{t}) \cdot \tau^2 \ . \end{aligned}$$

II. The general model.

1. Definition of the model, and transformation of the variables.

r independent random vectors are defined by

$$(1.1)\quad X_i = \begin{bmatrix} X_{i1} \\ \vdots \\ X_{in_i} \end{bmatrix} = F_i\Theta + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} U_i + \begin{bmatrix} V_{i1} \\ \vdots \\ V_{in_i} \end{bmatrix}, \quad i = 1,\dots,r.$$

The F_i's are known $n_i \times f$ matrices with

$$R_g F_i = f_i$$

and $\Theta = \left[\Theta_1,\dots,\Theta_f\right]'$ is an unknown vector. The U_i's and V_{ij}'s are independent and normally distributed with means zero and variances τ^2 and σ^2 respectively. We want to test hypotheses about $\Delta = \frac{\tau^2}{\sigma^2}$. Let

$$X' = \left[X_1',\dots,X_r'\right],$$

$$F' = \left[F_1',\dots,F_r'\right],$$

$$V_i' = \left[V_{i1},\dots,V_{in_i}\right], \quad i = 1,\dots,r,$$

$$V' = \left[V_1',\dots,V_r'\right],$$

$$U' = \left[U_1,\dots,U_1,\dots,U_r,\dots,U_r\right],$$

where U has n_i elements equal to U_i, $i = 1,\dots,r$.

We assume

$$Rg\ F = f.$$

From (1.1)

$$X = F\Theta + U + V. \tag{1.2}$$

Define $Y_i = [Y_{i1},\dots,Y_{in_i}]'$ by

$$Y_i = P_iX_i\ , \qquad i = 1,\dots,r,$$

and let

$$P = \begin{bmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_r \end{bmatrix} .$$

Then $Y' = [Y_1',\dots,Y_r']$ is given by

$$Y = PX .$$

The P_i's (and thus P) are orthogonal matrices. Element (j,k) in P_i is denoted $p_{jk}^{(i)}$ and $p_{1k}^{(i)} = \frac{1}{\sqrt{n_i}}$. $k = 1,\dots,n_i$; $i = 1,\dots,r$. For simplicity we assume $[p_{11}^{(i)},\dots,p_{1n_i}^{(i)}] \in \mathcal{C}(F_i)$, $i = 1,\dots,r$, and the f_i first rows of P_i are chosen to span $\mathcal{C}(F_i)$. *)

. . . .

*) When $[p_{11}^{(i)},\dots,p_{1n_i}^{(i)}] \notin \mathcal{C}(F_i)$, the f_i+1 first rows of P_i are chosen to span a space that contains $\mathcal{C}(F_i)$.

Therefore $EY_{ij} = 0$, $j > f_i$, $i = 1,\ldots,r$. As in I, Section 1 we get that all Y_{ij} are independent and that

$$\left.\begin{aligned} \operatorname{Var} Y_{i1} &= (n_i\Delta+1)\sigma^2 \\ \operatorname{Var} Y_{ij} &= \sigma^2,\ j = 2,\ldots,n_i \end{aligned}\right\} \quad i = 1,\ldots,r$$

Furthermore $EY = PF\cdot\Theta$. Since P is non-singular, $Rg(PF) = f$.

Define the vector Z to consist of the Y_{ij}, $j = 1,\ldots,f_i$, $i = 1,\ldots,r$, so that all Y_{ij} not contained in Z are distributed like $N(0,\sigma^2)$. The matrix G is now defined through

$$EZ = G\Theta , \qquad Rg\ G = f.$$

Let

$$g = \sum_{i=1}^{r} f_i \qquad (n \geq g \geq f),$$

and thus there exists a $g \times g$ orthogonal matrix Q, where the first f rows span $\mathscr{L}(G)$. Now S_o and S_1 are given by

$$\begin{bmatrix} S_o \\ S_1 \end{bmatrix} = \begin{bmatrix} S_{01} \\ \vdots \\ S_{of} \\ S_{11} \\ \vdots \\ S_{1,g-f} \end{bmatrix} = QZ ,$$

and $S_2 = \left[S_{21}, \ldots, S_{2,n-g}\right]'$ consists of the elements Y_{ij}, $j = f_i+1, \ldots, n_i$, $i = 1, \ldots, r$. Then

$$(1.3) \qquad \begin{aligned} ES_0 &= H \cdot \Theta \ , \quad \mathrm{Rg}\ H = f, \\ ES_1 &= 0 \ , \quad ES_2 = 0, \end{aligned}$$

where H consists of the f first rows of QG. S_1 and S_2 are independent, and

$$(1.4) \qquad \begin{aligned} \Sigma(S_1) &= (Q^*DQ^{*\prime}\Delta + I)\sigma^2 \ , \\ \Sigma(S_2) &= I\sigma^2 \ , \end{aligned}$$

where Q^* consists of the $g-f$ last rows of Q, and D is a diagonal matrix, with a n_i or zero as diagonal element.

Finally we make the transformation

$$(1.5) \qquad R = KS_1 .$$

Here K is a nonsingular matrix which makes $KQ^*DQ^{*\prime}K' = \lambda$, where λ is a diagonal matrix with the eigenvalues of $Q^*DQ^{*\prime}$ (λ_i, $i = 1, \ldots, g-f$) as diagonal elements. Since $Q^*DQ^{*\prime}$ is positive semidefinite, all $\lambda_i \geq 0$. We let $l = \mathrm{Rg}(Q^*DQ^{*\prime}) = \mathrm{Rg}\ \lambda$, and number the λ_i's to make $\lambda_i > 0$, $i = 1, \ldots, l$, and $\lambda_i = 0$, $i = l+1, \ldots, g-f$.

We have $\Sigma(R) = (\lambda\Delta + I)\sigma^2$.

2. Optimum invariant tests for the variance ratio.

We consider the hypotheses

$$(2.1)\quad \begin{array}{llll} H_1 : \Delta \leq \Delta_0 & \text{against} & \Delta = \Delta_1 & (\Delta_1 > \Delta_0) \\ H_2 : \Delta \leq \Delta_0 & \text{against} & \Delta \geq \Delta_1 & \\ H\ \ : \Delta \leq \Delta_0 & \text{against} & \Delta > \Delta_0 & \end{array}$$

These hypothesis problems are invariant under the following group G_1 of transformations

$$(2.2)\quad \begin{array}{l} S^*_{oi} = S_{oi} + c_i, \quad -\infty < c_i < \infty, \qquad i = 1,\dots,f \\ S^*_1 = S_1, \\ S^*_2 = S_2. \end{array}$$

$S = \begin{bmatrix} R \\ S_2 \end{bmatrix}$ (or equivalently $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$) is a maximal invariant with respect to G_1. The probability density is

$$(2.3)\quad k(\sigma,\Delta)\exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{g-f}\frac{1}{\lambda_i{}^{\Delta+1}}R_i^2 + \sum_{j=1}^{n-g} S_{2j}^2\right]\right\}$$

where $k(\sigma,\Delta)$ is a constant depending on σ and Δ only. We may write (2.3) as

$$(2.4)\qquad a(s,\Delta,\sigma)\cdot b(u,\sigma,\Delta)$$

where

$$(2.5) \qquad a(s,\Delta,\sigma) = \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{g-f}\left(\frac{1}{\lambda_i\Delta+1}-\frac{1}{\lambda_i\Delta_0+1}\right)R_i^2\right\},$$

and where

$$U = \sum_{i=1}^{g-f}\frac{1}{\lambda_i\Delta_0+1}R_i^2 + \sum_{j=1}^{n-g}S_{2j}^2$$

is sufficient and complete when $\Delta = \Delta_0$.

We proceed as in part I to find optimum invariant tests for the hypotheses (2.1). The UMPI test for H_1 have the rejection region

$$(2.6) \qquad W_1 = \frac{\sum_{i=1}^{\ell}\left(\frac{1}{\lambda_i\Delta_0+1}-\frac{1}{\lambda_i\Delta_1+1}\right)R_i^2}{U} > c_1 ,$$

and is also a maximin test for H_2.

Letting $\Delta_1 \to \infty$ we get the test φ_2 for H, with rejection region

$$(2.7) \qquad W_2 = \frac{\sum_{i=1}^{\ell}\frac{R_i^2}{\Delta_0\lambda_i+1}}{\sum_{i=\ell+1}^{g-f}R_i^2 + \sum_{j=1}^{n-g}S_{2j}^2} > c_2 .$$

The LMPI test for H is φ_3, with the rejection region

$$(2.8) \qquad W_3 = \frac{\sum_{i=1}^{\ell}\frac{\lambda_i}{(\Delta_0\lambda_i+1)^2}R_i^2}{U} > c_3$$

Power functions of φ_1, φ_2 and φ_3 are easily established (see part I), and they are seen to increase in Δ.

For the distribution of S to involve Δ, we must have $l > 0$. (When $l = 0$ a UMPI test which rejects with probability ϵ for all X is found.) In the next section the condition $l > 0$ will be expressed in another way.

3. Determination of the maximal invariant.

Again we consider the determination of the maximal invariant, and also give some examples. Now let

$$EX_{ij} = \sum_{p=1}^{a_i} \alpha_i^{(p)} s_{ij}^{(p)} + \sum_{p=1}^{b} \beta^{(p)} t_{ij}^{(p)} , \tag{3.1}$$

where all $\alpha_i^{(p)}$'s and $\beta^{(p)}$'s are unknown parameters (which implies $f = \sum_{i=1}^{r} a_i + b$), and all $\beta^{(p)}$'s are elements of the expectation in at least two groups. $s_{ij}^{(p)}$ and $t_{ij}^{(p)}$ are given constants. We let

$$\alpha_i = \left[\alpha_i^{(1)}, \dots, \alpha_i^{(a_i)}\right]' , \quad i = 1, \dots, r,$$
$$\beta = \left[\beta^{(1)}, \dots, \beta^{(b)}\right]' ,$$

and have

$$EX_i = A_i \alpha_i + B_i \beta , \qquad i = 1, \dots, r, \tag{3.2}$$

where the elements of the matrices A_i and B_i are given by (3.1). The F_i's, F and Θ in Section 1 are now given by

$$
(3.3) \quad F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_r \end{bmatrix} = \left[\begin{array}{cccc|c} A_1 & & & 0 & B_1 \\ & A_2 & & & B_2 \\ & & \ddots & & \vdots \\ 0 & & & A_r & B_r \end{array}\right] = \left[A \,\vdots\, B\right],
$$

$$
\theta' = [\alpha_1', \ldots, \alpha_r', \beta'] \quad .
$$

(3.3) also constitutes a definition of A and B.

Suppose

$$
\operatorname{Rg} A_i = a_i ,
$$

$$
\operatorname{Rg} B = b ,
$$

and define

$$
\operatorname{Rg} B_i = b_i ,
$$

$$
\operatorname{Rg} A = \sum_{i=1}^{r} a_i = a.
$$

Let

$$
Y_i = [Y_{i1}, \ldots, Y_{in_i}]' = P_i X_i , \quad i = 1, \ldots, r,
$$

where the a_i first rows of the orthogonal matrix P_i now are chosen to span $\mathcal{C}(A_i)$. Thus Y_{ij}, $j = 1, \ldots, a_i$, $i = 1, \ldots, r$ will be elements of S_o (i.e. not belong to the maximal invariant).

For those i where $(1, \ldots, 1) \in \mathcal{C}(A_i)$ we have $\sum_{m=1}^{n_i} p_{km}^{(i)} = 0$ for $k > a_i$, and all Y_{ij}, $j > a_i$ will be independently distributed as $N(0, \sigma^2)$, (and thus belong to

S_2). Groups of this type evidently give us no trouble, and to simplify the notation we assume

$$(3.4) \qquad (1,\ldots,1) \notin \mathscr{C}(A_i), \quad \text{all } i.$$

(For the distribution of the maximal invariant to involve Δ, (3.4) <u>must</u> be satisfied for at least one i.)

As row nr. a_i+1 we choose the component of $(1,\ldots,1)$ which is orthogonal to the first a_i rows.

As in Section 1 we also assume

$$(3.5) \qquad (1,\ldots,1) \in \mathscr{C}(F_i), \quad \text{all } i.$$

The f_i first rows of P_i are chosen to span $\mathscr{C}(F_i)$.– All Y_{ij}, $j = a_i+1,\ldots,n_i$, $i = 1,\ldots,r$ are now independent and

$$(3.6) \qquad \left.\begin{aligned} \operatorname{Var} Y_{i,a_i+1} &= (h_i^2\Delta+1)\sigma^2 \\ \operatorname{Var} Y_{i,j} &= \sigma^2 , \ j > a_i+1 \end{aligned}\right\} \ i = 1,\ldots,r,$$

where $h_i = \sum_{k=1}^{n_i} p_{a_i+1,k}^{(i)}$. (From (3.4) we have $h_i \neq 0$.) Furthermore $EY_{ij} = 0$, $j > f_i$.–Now define Z_1 to consist of Y_{i,a_i+1}, $i = 1,\ldots,r$ and Z_2 to consist of Y_{ij}, $j = a_i+2,\ldots,f_i$, $i = 1,\ldots,r$. EZ_p lies in a space with dimension d_p, $p = 1,2$.

Define T_1 and T_2 (with elements T_{1j} and T_{2j} respectively) by

$$(3.7) \qquad T_p = Q_p Z_p, \quad p = 1,2,$$

where Q_1 and Q_2 are orthogonal matrices chosen to make

$$ET_{1j} = 0 \ , \quad j > d_1,$$
$$ET_{2j} = 0 \ , \quad j > d_2.$$

Then we can find a nonsingular matrix N, which by defining $M = \left[M_1, \ldots, M_{d_1+d_2}\right]'$ through

$$(3.8) \qquad M = N \begin{bmatrix} T_{11} \\ \vdots \\ T_{1d_1} \\ T_{21} \\ \vdots \\ T_{2d_2} \end{bmatrix} ,$$

makes

$$EM_j = 0, \ j > f-a = b.$$

Define S_o to consist of the elements Y_{ij}, $j = 1,\ldots,a_i$, $i = 1,\ldots,r$ and M_j, $j = 1,\ldots,f-a$, S_1 to consist of T_{1j}, $j = d_1+1,\ldots,r$ and M_j, $j = f-a+1,\ldots,d_1+d_2$, and S_2 to consist of Y_{ij}, $j = f_i+1,\ldots,n_i$, $i = 1,\ldots,r$, and T_{2j}, $j = d_2+1,\ldots,f-a-r$. Then $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ is a maximal invariant when adding constants to the elements of S_o, and has distribution like that described in Section 1.

With this (new) definition of S_1 and S_2 a necessary and sufficient condition for S to involve Δ, is that S_1 contains at least one element. That is, we must have

$$(3.9) \qquad r+d_2 > b.$$

(Thus it is sufficient that $r > b$.)

Example 1. The model treated in part I was

$$X_{ij} = \mu + \beta t_{ij} + U_i + V_{ij} .$$

Here $a_i = 0$ and

$$B_i = \begin{bmatrix} 1 & t_{i1} \\ \vdots & \vdots \\ 1 & t_{in_i} \end{bmatrix} .$$

(3.4) and (3.5) are satisfied, and except for those cases treated in I, Section 7 we have (since $n_i \geq 2$)

$$f_i = b_i = 2, \; f = b = 2,$$

$$d_1 = 2, \quad d_2 = 1.$$

Since $r \geq 2$, (3.9) is satisfied. The matrix N and the vector M (which was not introduced in part I) are given by

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \sqrt{\frac{M^*}{M}} \\ 0 & 1 & -\sqrt{\frac{M^*}{M}} \end{bmatrix} \qquad M = \begin{bmatrix} S_{o1} \\ S_{o2} \\ S_{11} \end{bmatrix}$$

When $t_{i1} = \dots = t_{in_i}$, $i = 1,\dots,r$, we have $b_i = 1$, $b = 2$ and $d_1 = 2$, $d_2 = 0$.

When $\bar{t}_{1.} = \dots = \bar{t}_{r.}$, we have $b_i = b = 2$, $d_1 = d_2 = 1$.

When all t_{ij}'s are equal, we have $b_i = b = 1$ and $d_1 = 1$, $d_2 = 0$.

These three special cases are examples of the situation $a_i = 0$ (all i) and $d_1+d_2 = b$. Then all elements of S_1 have the same variance, when the model is balanced; and so there exists a UMPI test for H (see I, Section 7). In this situation X has an exponential distribution, and the UMPI test coincides with a UMP unbiased one (see [2], [5]).

Example 2. We consider the model

$$X_{ij} = \mu + \alpha_i s_{ij} + U_i + V_{ij}, \; j = 1,\dots,n_i, \; i = 1,\dots,r.$$

Here $n_i \geq 2$, $r \geq 2$, and the s_{ij} are chosen so that (3.4) is satisfied. Thus

$$a_i = 1, \quad a = r,$$

$$b_i = b = 1$$

$$f_i = 2, \quad f = r+1$$

The first two lines of P_i are given by

$$\left.\begin{aligned} p_{1k}^{(i)} &= \frac{s_{ik}}{\left(\sum_j s_{ij}^2\right)^{\frac{1}{2}}} \\ p_{2k}^{(i)} &= \left[1 - \frac{s_{ik}}{\sum_j s_{ij}^2} \; n_i \bar{s}_{i.}\right] k_i \end{aligned}\right\} \quad k = 1,\dots,n_i$$

where k_i is a norming constant.

Y_{ij}, $j = 2,\dots,n_i$, $i = 1,\dots,r$ are independent and

$$EY_{i2} = k_i\left[n_i - \frac{\left(\sum_j s_{ij}\right)^2}{\sum_j s_{ij}^2}\right]\mu = h_i \cdot \mu,$$

$$\text{Var } Y_{i2} = (h_i^2 \cdot \Delta + 1)\sigma^2,$$

$$\left.\begin{aligned} EY_{ij} &= 0 \\ \text{Var } Y_{ij} &= \sigma^2 \end{aligned}\right\} \quad j = 3,\dots,n_i,$$

$Z_1 = \left[Y_{12},\ldots,Y_{r2} \right]'$ is transformed into T_1, which has $r-1$ elements with mean zero. These $r-1$ elements constitute S_1, whilst Y_{ij}, $j = 3,\ldots,n_i$, $i = 1,\ldots,r$ constitute S_2.

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